Strong one-dimensional localization in systems with statistically rough boundaries

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It is shown that multiple scattering from surface roughness causes localization of waves in a one-mode waveguide and of electronic states in a thin film. The effect of the boundary roughness on the spectral properties of such systems is studied.

One of the most important problems of the theories of disordered condensed systems and of wave propagation in media with fluctuating parameters is how to account consistently for the interference of multiply scattered fields. In the last decade, several monographs have appeared placing considerable emphasis on the study of multiple scatterings by fluctuations of the potential describing bulk inhomogeneities.[1-4] In calculations of fields above slightly rough surfaces, multiple scatterings are usually disregarded. This largely stems from the fact that in most practically interesting cases (e.g., radio-wave propagation over the sea or electron propagation in bulk samples), a wave (or a particle), after having been once scattered by a rough surface, propagates in a homogeneous medium without further interaction with the scattering boundary.

If the characteristic roughness height is much smaller than the wavelength, then satisfactory results are provided by the first Born approximation of perturbation theory; in the case of arbitrarily high but smooth roughness, the Kirchoff approximation suffices. These methods permitted satisfactory explanation of a lot of experimental facts;[5,6] they, however, prove inadequate when multiple scattering gives a predominant contribution. This is the case for scattering in the direction opposite that of wave incidence (backscattering enhancement[7] and weak localization[8]), for surface wave propagation,[9,10] and also for wave propagation in media bounded by two or more reflecting surfaces. Such effects are observed in waveguide communication lines and natural waveguides (the Earth-ionosphere system, the tropospheric waveguide, the underwater sound channel, etc.), in which the role of the second boundary is played by regular refraction due to the nonmonotonic dependence of the modified refractive index on the coordinates.[11] Finally, in solid-state samples, the electron-returning factor may be a magnetic field,[12,13] or the other boundary of a thin film.[14,15] In all such systems, a wave (a particle) scattered by roughness, after having been reflected by the second boundary or owing to refraction (or under the action of a magnetic field), is scattered by the roughness again, and so on. Even perturbations that are much smaller than the wavelength, at long distances, because of the accumulation, produce essential changes in the spatial-field distribution.

For taking account of multiple scattering from random surfaces, a diagram technique was developed[16] which yielded approximate closed integro-differential equations for the first two moments of the field of a point source in a nonregular waveguide[17] and for the one- and two-particle Green functions of conduction electrons in magnetic field.[12,15] The Dyson equation for the average field (the coherent component) can be solved exactly. However, the equation for the field correlation function can be reduced to the radiation-transfer equation only in the simplest, so-called "ladder" approximation. Its solution, in fact, amounts to summation of an infinite number of reflections, the diffraction being, however, taken into account only in the description of a single-scattering event and contributions of multiple scattering being summed "over the intensity," i.e., disregarding interference. Therefore, the radiation-transfer equations are unsuitable where the main role in field formation belongs to scattering in the opposite direction, which leads to backscattering enhancement, to weak localization in media with three-dimensional inhomogeneities, and to strong localization in one-dimensional disordered systems.

These effects are most prominent in waveguides (or thin films) with the transverse dimensions such that there exists only one mode, i.e., only the two normal waves

$$\xi_{\pm}(\mathbf{r})=\chi(x,z;q_x,q_a)\exp(\pm i\kappa y), \quad \mathrm{Im}\kappa=0 \tag{1}$$

can propagate, and thus scattering from inhomogeneities can occur only either forwards ($\kappa\rightarrow\kappa$) or backwards ($\kappa\rightarrow-\kappa$).

Here y is the coordinate along the waveguide axis, x and z are the coordinates in the cross section, κ is the longitudinal wave number, q_x and q_z are the transverse wave numbers, so that $q_x^2+q_2^2+\kappa^2=(2\pi/\lambda)^2=k^2$ (λ being the radiation wavelength), and $\chi(x,z;q_x,q_z)$ is the transverse eigenfunction, satisfying the equation

$$(\nabla_\perp^2+k_\perp^2)\chi(\mathbf{r}_\perp)=0\ , \tag{2}$$

$$k_\perp^2=q_x^2+q_z^2, \quad \mathbf{r}_\perp=(x,z) \tag{3}$$

and the boundary condition at the waveguide walls, e.g., the Dirichlet condition.

Let an "unperturbed" waveguide be formed by an arbitrary cylindrical surface S. Normal deviations of a rough surface Σ from a smooth one S will be described by the random function $\xi(r_S)$,

$$\mathbf{r}_\Sigma=\mathbf{r}_S+\mathbf{N}\cdot\zeta(\mathbf{r}_s),\quad \mathbf{r}_\Sigma\in\Sigma,\quad \mathbf{r}_S\in S$$

(**N** is the normal to the mean surface S).

If the heights of these deviations are small compared with the radiation wavelength, the exact condition

$$\xi(\mathbf{r}_\Sigma)=0$$

can be expanded in a power series of $\zeta(\mathbf{r}_S)$ and thus extended to the smooth surface S:

$$\xi(\mathbf{r}_S)+\zeta(\mathbf{r}_S)\frac{\partial\xi(\mathbf{r})}{\partial N}\Bigg|_{\mathbf{r}=\mathbf{r}_S}=0\ . \tag{4}$$

By using the Green theorem, with consideration for relation (4), we can obtain for the field $\xi(\mathbf{r})$ in a waveguide with slightly rough walls:

$$\xi(\mathbf{r})=\chi(\mathbf{r}_\perp)\exp(-i\kappa y)$$
$$-\int dS'\,\zeta'(\mathbf{r}'_S)\frac{\partial\xi}{\partial N'}\chi(\mathbf{r}_\perp)\frac{\partial\chi}{\partial N'}\,\frac{\exp(i\kappa|y-y'|)}{2i\kappa}\ , \tag{5}$$

where dS' is an element of the surface on which the boundary condition (4) is defined, and over which the integration is performed.

The derivation of (5) is based on the following approximate expression for the Green function for an unperturbed one-mode waveguide:

$$G_0(\mathbf{r},\mathbf{r}')=\chi(\mathbf{r}_\perp)\chi(\mathbf{r}'_\perp)\frac{\exp(i\kappa|y-y'|)}{2i\kappa}\ .$$

It follows from Eq. (5) that

$$\varphi=\left[\frac{\partial\xi(\mathbf{r})}{\partial N}\right]_{\mathbf{r}=\mathbf{r}_S}\left[\frac{\partial\chi(\mathbf{r}_\perp)}{\partial N}\right]^{-1}_{\mathbf{r}=\mathbf{r}_S}$$

satisfies the one-dimensional integral equation:

$$\varphi(y)=\exp(-i\kappa y)$$
$$-\int_0^L dy'\,V(y')\frac{\exp(i\kappa|y-y'|)}{2i\kappa}\varphi(y')\ , \tag{6}$$

where

$$V(y)=\oint ds\,\zeta(s)(\partial\chi/\partial N)^2_{\mathbf{r}_\perp=\mathbf{r}_S,\ y=\text{const}} \tag{7}$$

L is the rough section length along the waveguide axis y.

Applying the operator $d^2/dy^2+\kappa^2$ to both sides of Eq. (6), we see that it is equivalent to the following:

$$d^2\varphi/dy^2+V(y)\varphi+\kappa^2\varphi=0\ , \tag{8}$$

Thus, the problem of finding the field in a waveguide with a statistically irregular boundary reduces to solving the one-dimensional Schrödinger Eq. (8) with a random potential, which has yet to be supplemented by the boundary conditions at the ends of the segment $[0,L]$.

This equation describes, as is known, the electron motion in a random external field $V(y)$. It is important that in the present case this field is not associated with bulk scatterers, but with surface inhomogeneities. Investigation of the dynamic and statistical properties of the solution of Eq. (8) is one of the central problems of the quantum theory of disordered condensed systems.[1] Reference 1 deals, however, with the so-called closed one-dimensional systems; that is, systems at whose boundaries 0 and L the following self-adjoint boundary conditions hold: $[\varphi(y)+C\varphi'(y)]|_{y=0,L}=0$, where $\mathrm{Im}C=0$, and ideal reflection then takes place at points 0 and L. In the case under discussion, at least on one side of the inhomogeneous region the solution at infinity must have the form of an outgoing wave (the radiation condition).

The spectral properties of such open one-dimensional disordered systems were studied in Ref. 18 for the case where $V(y)$ is a statistically homogeneous random function with correlation decreasing at infinity. It was shown that though the dispersion equation for the poles of the Green function of Eq. (8) has no real-valued roots, if the inhomogeneous region $[0,L]$ is sufficiently long, then there exist complex-valued solutions ε_n including an exponentially small imaginary part $\delta\varepsilon_n\propto\exp(-L/L_0)$. L_0 is the so-called localization length, inversely proportional to the Lyapunov exponent of Eq. (8). In our case it depends on the waveguide configuration. Thus, e.g., for a waveguide with a rectangular cross section with sides d_x and d_z, and a rough wall $x=0$, $0\le z\le d_z$, and $0\le y\le L$,

$$L_0=2\pi d_x(kd_x/\pi)^3(d_x/\pi\sigma)^2[2kl\tilde{W}(2kl)]^{-1}\ , \tag{9}$$

where

$$\tilde{W}(2kl)=\frac{4}{d_z^2}\int_0^{d_z}dz\int_0^{d_z}dz'\sin^2\left[\frac{\pi z}{d_z}\right]\sin^2\left[\frac{\pi z'}{d_z}\right]\times W(2kl;z,z')\ .$$

$W(2kl;z,z')$ is the dimensionless Fourier transform in the longitudinal coordinate y of the roughness correlation function

$$\langle\zeta(y,z)\zeta(y',z')\rangle=\sigma^2 w(|y-y'|/l;z,z')\ ,$$

where σ^2 is the rms roughness height, and l is the roughness correlation radius. Thus, though the spectrum is formally continuous in this case, it still becomes quasidiscrete in a disordered open system, viz., the spectral density represents a set of sharp peaks at the real values $E_n\approx(\pi n/L)^2$, the characteristic distances between them, ΔE_n, being of the order of $\Delta E_n\approx 2\pi^2 n/L^2$, and their widths ΔE_n being exponentially small in the parameter L/L_0:[18]

$$\delta E_n=2\,\delta\varepsilon_n\approx\Delta E_n\exp[-L/L_0(E_n)]\ . \tag{10}$$

The quantities ε_n are random and are determined by the configuration of a particular realization. Their complete set represents a typical mesoscopic characteristic of each realization, such as, e.g., the dependences on the magnetic field in mesoscopic conductors (what are called "magnetofingerprints").

The wave function φ_n corresponding to the values ε_n describe waves similar to those known in quantum mechanics as quasistationary decay states.[19] It is important that in the present case these states arise not in a potential well, as in the case, e.g., of a nucleus, nor because of scattering from bulk inhomogeneities, but from surface roughness. A radical difference from the usual "tunneling" quasistationary states is that in this case the functions $\varphi(y,E_n)$ are localized in a sufficiently narrow region of space, i.e., their envelopes are essentially different from zero in the interval $L_0(E_n)$, near randomly localization centers y_n: $\varphi(y,E_n)\propto\exp(-|y-y'|/L_0)$. In Ref. 18 this statement was proved for the problem in the semiaxis.

Thus, a sufficiently long rough section of the surface behaves as efficient walls, i.e., it locks the radiation (or a quantum particle) in the direction along the y axis of the waveguide. As a result, in a waveguide formed by the plane $z=d$ and a random (plane in the mean) surface $z=\zeta(y)$ ($\langle\zeta\rangle=0$ and $\langle\zeta^2\rangle=\sigma^2$), the field of a point source, for $[kd/\pi]=1$ and $k\sigma\ll 1$, is as follows:

$$G(\mathbf{r},\mathbf{r}')\approx\sum_n A_n \sin\left[\frac{\pi z}{d_z}\right]\sin\left[\frac{\pi z'}{d_z}\right] \times\varphi(y,E_n)\varphi(y',E_n)\exp(i\kappa_n|x-x'|)\ , \qquad (11)$$

where r′ is the point at which the source is, and

$$\kappa_n=[k^2-(\pi/d_z)^2-\varepsilon_n]^{1/2}\ .$$

Consequently, one-dimensional (oriented along the x axis) roughness, if appearing in a plane waveguide, makes waves plane (not cylindric, as was the case for $\zeta=0$) and quasihomogeneous, which thus propagate along the x axis with an exponentially small attenuation.

A waveguide with a closed cross section (e.g., rectangular with sides d_x and d_z), its boundaries being nonregular, becomes, because of wave localization along its axis, a kind of resonator whose natural frequencies ω_n are specified by the relation

$$\left[\frac{\omega_n}{c}\right]^2=\left[\frac{\pi}{d_x}\right]^2+\left[\frac{\pi}{d_z}\right]^2+E_n\ ,$$

and the Q factor is determined by the imaginary part of ε_n i.e., it depends on the roughness parameters and the nonregular section length L (Ref. 18 predicted the existence of such a resonator in a randomly layered medium).

As was shown above, scattering from surface inhomogeneities results in wave and particle localization in one-mode systems. This manifests itself not only in rearrangement of the scattering states, but also in peculiar behavior of the transmission coefficient $t_L(\kappa)$ and the reflection coefficient $r_L(\kappa)$ of a rough section of length L. The scattering problem for Eq. (8) was well studied in Refs. 1, 3, and 18. It was shown, in particular, that at typical realizations the transparence T, i.e., the square of the transmission-coefficient modulus, exponentially decreases as the inhomogeneous segment length increases with a decrement equal to the inverse localization length:

$$T=|t_L(\kappa)|^2\propto\exp(-L/L_0)\ , \qquad (12)$$

and the squared reflection-coefficient modulus differs from unity by an exponentially small amount. The average transparence is also exponentially small as a function of the parameter L/L_0. Thus, e.g., for a waveguide with a rectangular cross section, the distribution function of $u=2/T-1$, is

$$P(u,L)=(8\pi)^{1/2}\left[\frac{L_0}{L}\right]^{3/2}\exp\left[-\frac{L}{4L_0}\right] \times\int_{\mathrm{arccosh}u}^{\infty}dx\frac{x}{(\cosh x-u)^{1/2}}\exp\left[-\frac{L_0}{4L}x^2\right] \qquad (13)$$

and the average value $\langle T\rangle$ is

$$\langle T\rangle=\frac{\pi^{5/2}}{2}\left[\frac{L_0}{L}\right]^{3/2}\exp\left[-\frac{L}{4L_0}\right]\ . \qquad (14)$$

Note that summation of the intensities of multiply-scattered fields neglecting interference (the radiation-transfer approximation) results in a power-law asymptotic behavior of the average transparence for large L: $\langle T\rangle\propto 1/L$.[2,17]

Comparison of formulas (12) and (14) shows that the average transparence is materially higher than that at typical realizations (their decrements differ by a factor of 4). This is so because the main contribution to $\langle T\rangle$ is given not by typical but by resonance realizations of exponentially low probability at which the transparence is almost complete ($T\approx 1$), and these realizations are representative for $\langle T\rangle$. In Ref. 18 it was demonstrated that in the case of ideal transparence, resonance scattering states have, inside the disordered segment, a clearly defined localized character.

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