Nonperturbative results for the spectrum of surface-disordered waveguides

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We calculated the spectrum of normal scalar waves in a planar waveguide with absolutely soft randomly rough boundaries. Our approach is beyond perturbation theories in the roughness heights and slopes and is based instead on the exact boundary scattering potential. The spectrum is proved to be a nearly real nonanalytic function of the dispersion ζ^2 of the roughness heights (with square-root singularity) as $\zeta^2 \to 0$. The opposite case of large boundary defects is summarized. © 1998 Optical Society of America

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In finite systems long-distance (waveguide) signal propagation is caused by multiple reflections of the signal from opposite lateral boundaries. If the boundaries are irregular, each of the reflections is accompanied by noncoherent scattering of the wave. The multiple successive scattering events lead to substantial dephasing and attenuation of the signal.

As far as we know this effect was first consistently treated in Refs. 1 and 2. This simple and physically clear approach was based on the perturbation theory in the squared rms height ζ of the boundary roughness and therefore required a small enough ζ, i.e., $(k_z\zeta)^2 \ll 1$, $(\zeta/R_c)^2 \ll 1$, and $k\zeta^2/R_c \ll 1$. Here k_z is the transverse (normal to the waveguiding direction) component of the wave vector $\mathbf{k}$ and R_c is the mean length of the boundary defects. More recently[3–6] the theory of wave scattering from statistically rough surfaces was extended to arbitrary values of the Rayleigh parameter $(k_z\zeta)^2$ through the expansion in small roughness slopes $[(\zeta/R_c)^2 \ll 1]$.

In this Letter we put forward an approach that is nonperturbative in the roughness heights and slopes. It is based instead on the exploitation of the exact boundary scattering operator. Because of this we extend the waveguide theory to the quite-general conditions of weak scattering [see relations (8), below], which are much less restrictive than those of the above approximations. The most impressive advantage of our method is that it leads to new physical results even for the region of small heights ζ, in which the waveguide propagation is believed to be well studied. The most surprising result is a nonanalytic (square-root) dependence of the waveguide spectrum on the dispersion ζ^2 of the roughness heights. This nonanalyticity in principle cannot be derived perturbatively, which means that the exact randomly rough boundary can never be reduced to a smooth random-impedance boundary.

We consider a two-dimensional (planar) strip confined to a region of the xz plane defined by $\xi(x) \le z \le d$, where $\xi(x)$ is a Gaussian-distributed random function of the longitudinal coordinate x characterized by

$$\langle \xi(x)\rangle = 0, \qquad \langle \xi(x)\xi(x')\rangle = \zeta^2 \mathcal{W}(|x - x'|)\,. \tag{1}$$

The angle brackets denote an average over the ensemble of random functions $\xi(x)$. The binary coefficient of correlation $\mathcal{W}(|x|)$ has unit amplitude and a typical width of R_c. The spatial distribution of a scalar wave field inside the strip is governed by the Helmholtz equation, and the temporal dependence is described by the factor $\exp(-i\omega t)$, $k = \omega/c$. Both boundaries $z = \xi(x)$ and $z = d$ are supposed to be absolutely soft, i.e., the field vanishes at these boundaries. We seek the averaged Green's function $\langle \mathcal{G}(x, x'; z, z')\rangle$ of this Dirichlet boundary-value problem.

The exact integral equation for $\mathcal{G}(x, x'; z, z')$ can be obtained through the use of Green's theorem:

$$\mathcal{G}(x,x';z,z') = \mathcal{G}_0(|x - x'|; z, z') + \int_{-\infty}^{\infty} \mathrm{d}x_s \mathrm{d}z_s \mathcal{G}_0(|x - x_s|; z, z_s)\hat{\Xi}(x_s, z_s)\mathcal{G}(x_s, x'; z_s, z')\,, \tag{2}$$

where $\hat{\Xi}(x_s, z_s)$ is the effective scattering operator,

$$\hat{\Xi}(x_s, z_s) = \delta[z_s - \xi(x_s)]\left[\frac{\partial}{\partial z_s} - \frac{\mathrm{d}\xi(x_s)}{\mathrm{d}x_s}\frac{\partial}{\partial x_s}\right], \tag{3}$$

and $\mathcal{G}_0(|x - x'|; z, z')$ is the Green's function for the ideal (smooth) strip with $\xi(x) \equiv 0$.

To perform averaging of Eq. (2) we apply the elegant technique derived in Ref. 7. As a result, we find the averaged Green's function:

$$\langle \mathcal{G}(x,x';z,z')\rangle \equiv \bar{\mathcal{G}}(|x - x'|; z, z') = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_x}{2\pi} \exp[ik_x(x - x')]\bar{G}(k_x; z, z')\,, \tag{4}$$

where longitudinal Fourier transform $\bar{G}(k_x; z, z')$ is given by

$$\bar{G}(k_x;\, z,\, z') \simeq \frac{G_0(k_x; z, z')}{1 - k_z \cot(k_z d) M(k_x)}\,. \tag{5}$$

Here $k_z = k_z(k_x) = (k^2 - {k_x}^2)^{1/2}$ and $G_0(k_x; z, z')$ is the Fourier transform [similar to Eq. (4)] of $\mathcal{G}_0(|x - x'|; z, z')$. The self-energy $M(k_x)$ is defined by the binary correlator of the scattering operator (3):

$$M(k_x) = \int_{-\infty}^{\infty} \mathrm{d}z_s \mathrm{d}z_s' \mathrm{d}x_s \, \frac{\sin(k_z z_s)}{k_z} \exp(-ik_x x_s) \langle \hat{\Xi}(x_s, z_s) \times \mathcal{G}_0(|x_s - x_s'|; z_s, z_s') \hat{\Xi}(x_s', z_s') \rangle \exp(ik_x x_s') \, \frac{\sin(k_z z_s')}{k_z} \, . \tag{6}$$

By equating the denominator of relation (5) to zero, we obtain the dispersion equation for the rough-bounded strip whose solution in the lowest (linear) order in $M(k_x)$ is $k_x = k_n + \delta k_n$. Here $k_n = [k^2 - (\pi n/d)^2]^{1/2}$ is the unperturbed longitudinal wave number of an nth propagating normal mode and δk_n is a complex modification to k_n that is caused by wave scattering from the irregular boundary:

$$\delta k_n = \gamma_n + i(2L_n)^{-1} = -M(k_n)(\pi n/d)^2 / k_n d \, . \tag{7}$$

The real part γ_n of δk_n is responsible for variation of the phase velocity (for dephasing), and L_n is the attenuation length for the nth mode.

In fact, a form of the solution given by relations (5) and (7) is common and well known.[2,7] Our improvement lies in the self-energy $M(k_x)$ [Eq. (6)]. This equation is nothing but a first nonvanishing (quadratic) term in an expansion of $M(k_x)$ in powers of the exact scattering operator [Eq. (3)]. We stress that such an approximation is essentially different from and is much more general than an extensively exploited first term in expansions of $M(k_x)$ in powers of the dispersion ζ^2 or squared slopes $(\zeta/R_c)^2$.

To find the domain of validity for Eqs. (6) and (7) one can use the ideas proposed in Ref. 8. We have proved that this domain coincides with the two natural requirements of weak wave scattering, which are equations themselves with respect to the external dimensionless parameters $(k\zeta)^2$, kR_c, kd/π, and n:

$$|\delta k_n| \Lambda_n \ll 1 \, , \qquad |\delta k_n| \tilde{R}_c \ll 1 \, . \tag{8}$$

The first of these relations implies the smallness of the complex phase shift over the distance $\Lambda_n = 2k_n d/(\pi n/d)$ passed by an nth mode between two successive reflections from the rough boundary. This condition also ensures the smallness of δk_n in comparison with k_n, because the inequality $k_n \Lambda_n \gtrsim 1$ always holds. The second of relations (8) indicates that the phase shift must remain small over the typical variation scale $\tilde{R}_c$ (effective correlation radius) of the boundary scattering potential. Obviously this is the necessary and sufficient condition for correctness of statistical averaging over $\xi(x)$.

We next calculate the correlator and the integrals over z_s and z_s' in Eq. (6). Then we substitute the result into Eq. (7) and extract the real γ_n and the imaginary $(2L_n)^{-1}$ parts of δk_n. Finally, we get explicit formulas for γ_n and L_n:

$$\gamma_n = \frac{\zeta^2}{2} \frac{(\pi n/d)^2}{k_n d} \sum_{n'=1}^{n_d} \frac{(\pi n'/d)^2}{k_{n'} d} [\widetilde{W}_S(k_n, k_{n'}) - \widetilde{W}_S(k_n, -k_{n'})] - \frac{(\pi n/d)^2}{k_n d} M_2(k_n) + 2\zeta^2 \frac{(\pi n/d)^2}{k_n d} \times \sum_{n'=n_d+1}^{\infty} \frac{(\pi n'/d)^2}{|k_{n'}| d} \int_0^{\infty} \mathrm{d}x \exp(-|k_{n'}| x) \times \mathrm{Re}[\exp(-ik_n x) \widetilde{\mathcal{W}}(k_n, i|k_{n'}|; x)] \, , \tag{9}$$

$$L_n^{-1} = \zeta^2 \frac{(\pi n/d)^2}{k_n d} \sum_{n'=1}^{n_d} \frac{(\pi n'/d)^2}{k_{n'} d} [\widetilde{W}_C(k_n, k_{n'}) + \widetilde{W}_C(k_n, -k_{n'})] \, . \tag{10}$$

Here integer $n_d = (kd/\pi)$ is the number of propagating normal modes in the smooth strip. The function $\widetilde{\mathcal{W}}(k_x, q_x; x)$ is the generalized coefficient of correlation [$\widetilde{\mathcal{W}}(k_x, q_x; x) \simeq \mathcal{W}(|x|)$ as $\zeta^2 \to 0$]:

$$\widetilde{\mathcal{W}}(k_x, q_x; x) = (4k_z q_z \zeta^2)^{-1} \times \left\{ \left[(k_z + q_z)^2 + (k_z + q_z)(k_x - q_x)\left(\frac{k_x}{k_z} - \frac{q_x}{q_z}\right) - (k_x - q_x)^2 \frac{k_x q_x}{k_z q_z} \right] S(k_z + q_z, k_z + q_z; x) - \left[(k_z - q_z)^2 + (k_z - q_z)(k_x - q_x)\left(\frac{k_x}{k_z} + \frac{q_x}{q_z}\right) + (k_x - q_x)^2 \frac{k_x q_x}{k_z q_z} \right] S(k_z - q_z, k_z - q_z; x) + 2(k_x - q_x) \times \left[q_x \frac{k_z}{q_z} - k_x \frac{q_z}{k_z} + (k_x - q_x) \frac{k_x q_x}{k_z q_z} \right] \times S(k_z + q_z, k_z - q_z; x) \right\} , \tag{11}$$

$$S(t_1, t_2; x) = (t_1 t_2)^{-1} \sinh[t_1 t_2 \zeta^2 \mathcal{W}(|x|)] \times \exp[-({t_1}^2 + {t_2}^2)\zeta^2/2] \, , \tag{12}$$

where $q_z = k_z(q_x)$ and the functions $\widetilde{\mathcal{W}}_{S(C)}(k_x, q_x)$ stand for sine and cosine Fourier transforms of $\widetilde{\mathcal{W}}(k_x, q_x; x)$, respectively. The component $M_2(k_x)$ of the self-energy is given by

$$M_2(k_x) = \frac{k^2}{2{k_z}^2 d} \sum_{n'=1}^{\infty} \left\{ 2S(k_z + \pi n'/d, k_z - \pi n'/d; 0) - S(k_z + \pi n'/d, k_z + \pi n'/d; 0) - S(k_z - \pi n'/d, k_z - \pi n'/d; 0) - \frac{k_z}{k^2} \frac{\pi n'}{d} [S(k_z + \pi n'/d, k_z + \pi n'/d; 0) - S(k_z - \pi n'/d, k_z - \pi n'/d; 0)] \right\} \cdot \tag{13}$$

An essential distinction between γ_n and L_n^{-1} is that the latter is formed by scattering of a given nth propagating mode into propagating modes with $n' \le n_d$ only, whereas the former has a much more complicated structure owing to contributions of both propagating and evanescent ($n' > n_d$) modes. This feature is the basis for surprising properties of γ_n.

Next we discuss the results. We first consider relatively simple and widely used limiting case of small boundary perturbations,

$$(k\zeta)^2 \ll 1\,. \tag{14}$$

Here we simply reduce Eq. (10) for L_n to the standard result from Refs. 1 and 2 by replacing $\widetilde{\mathcal{W}}(k_x, q_x; x)$ with $\mathcal{W}(|x|)$. However, γ_n [Eq. (9)] shows unconventional behavior. The reason for this is that the last two terms of Eq. (9) are mainly formed by those evanescent modes whose normal wavelengths $(\pi n'/d)^1$ are of the order of the roughness height ζ. Each of these resonant modes makes a contribution to γ_n that is proportional to ζ^2, and the number n' of these modes is $\sim d/\zeta \gg n_d$, i.e., it is inversely proportional to ζ. These two properties yield a linear dependence of γ_n [Eq. (9)] on the roughness height ζ:

$$\gamma_n \sim \zeta(\pi n/d)^2/k_n d\,. \tag{15}$$

Relation (15) is the main result of this Letter. It leads to the following significant conclusions:

1. Since $L_n^{-1} \propto \zeta^2$ and $\gamma_n \propto \zeta$ as $\zeta^2 \to 0$, $\gamma_n \gg L_n^{-1}$, and hence the entire spectrum shift δk_n [Eq. (7)] turns out to be nearly real, i.e., $\delta k_n \simeq \gamma_n \propto \zeta$. This means that a signal propagating through a weakly corrugated waveguide is dephased over much shorter distances than its initial amplitude is damped.
2. From relation (15) and conclusion 1 it follows that δk_n is a nonanalytic (square-root) function of the dispersion ζ^2: $\delta k_n \propto (\zeta^2)^{1/2}$.
3. Usually it is believed that condition (14) is sufficient to enable one to infer that any long-wave normal mode with $(k_z\zeta)^2 \ll 1$ [i.e., with $(\pi n/d)^{-1} \gg \zeta$] is scattered mainly into the long-wave modes as well, $(\pi n'/d)^1 \gg \zeta$. This assumption allows immediate replacement of the exact Dirichlet boundary condition, formulated on a randomly rough boundary, with an approximate impedance condition formulated on the averaged (deterministic) boundary $z = 0$ with random impedance $\xi(x)$. We have proved that such a reduction is groundless, because the resonant evanescent modes with $(\pi n'/d)^{-1} \sim \zeta$ dominate. Thus the problem of wave propagation through a waveguide with an absolutely soft random boundary cannot be reduced to that with a smooth random-impedance boundary, even for arbitrarily weak perturbations.

An important step in analyzing relation (14) is to obtain the explicit weak-scattering condition. According to relations (8) and (15), we find that relation (8) can be rewritten as

$$(k_z\zeta)^2 = (\pi n/d)^2\zeta^2 \ll \min[1, (\Lambda_n/R_c)^2]\,. \tag{16}$$

This inequality is automatically satisfied within relation (14) if successive reflections of an nth mode from the rough boundary are not correlated[1,2] ($R_c \ll \Lambda_n$). However, if the correlations are strong ($\Lambda_n \ll R_c$), then Eq. (16) supplements relation (14) and can become even more restrictive than relation (14). Note that the roughness slope ζ/R_c can far exceed unity within the limit of relation (14) and Eq. (16).

The situation with large boundary defects, when

$$(k\zeta)^2 \gg 1\,, \tag{17}$$

is much more diverse and complicated than the case of relation (14). For the most part it is analyzable only numerically. Here we list a few related results:

1. In contrast with the case of relation (14), the imaginary part of δk_n [Eq. (7)] may well compete with its real part. Moreover, the situation with $L_n^1 \gtrsim |\gamma_n|$ is rather typical.
2. The real spectrum shift γ_n reverses sign as $k\zeta$ reaches some threshold value $k\zeta \approx 1.5$–2.5, which is slightly dependent on the other parameters. So the large boundary defects [relation (17)] may not only decrease but also increase the phase velocity of a propagating wave.
3. As $k\zeta \lesssim 2$, δk_n is almost insensitive to the slope ζ/R_c and to the presence of the shadowing effect, which is controlled by the Fresnel parameter $k\zeta^2/R_c$. However, as $k\zeta \gtrsim 2$, weak-scattering conditions (8) are fulfilled only for slopes that are not too steep ($\zeta/R_c \lesssim 2$–3), whereas the strong shadowing effect is still allowed ($k\zeta^2/R_c \lesssim 8$–10).

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