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## Memory function versus binary correlator in additive Markov chains

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### Abstract

We study properties of the additive binary Markov chain with short and long-range correlations. A new approach is suggested that allows one to express global statistical properties of a binary chain in terms of the so-called memory function. The latter is directly connected with the pair correlator of a chain via the integral equation that is analyzed in great detail. To elucidate the relation between the memory function and pair correlator, some specific cases were considered that may have important applications in different fields.

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### 1. Introduction

Much attention has been recently paid to the anomalous transport in 1D systems with correlated disorder. In contrast with the well studied case of a white-noise potential, the problem of propagation of either electron or electromagnetic waves through samples with random potentials with short or long-range correlations, is a big theoretical challenge.

The fundamental significance of this problem is due to recent exciting findings revising a commonly accepted belief that any randomness in 1D structures results in the Anderson localization. In particular, it was shown [1–4] that specific long-range correlations in 1D random potentials give rise to an emergence of frequency windows of a very high and very low transparency of waves through such potentials. It was also found that the positions and widths of these windows can be controlled by the form of the binary correlator of a scattering potential. A quite simple method was suggested for constructing random potentials that result in any predefined frequency window of an almost perfect transparency.

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From the experimental viewpoint, many of the results may have a strong impact for the creation of a new class of electron nanodevices, optic fibers, acoustic and electromagnetic waveguides with selective transport properties. The predictions of the theory have been verified experimentally [5,6] by studying transport properties of a single-mode electromagnetic waveguide with point-like scatterers. The agreement between experimental results and theoretical predictions was found to be unexpectedly good. In particular, the predicted windows of a complete reflection alternated by those of a good transparency, were clearly observed, in spite of strong experimental imperfections.

A further development of the theory of correlated disorder is due to the application to the surface scattering of electromagnetic waves (or electrons). The problem of a wave propagation (both classical and quantum) through guiding systems with corrugated surfaces has long history and still remains a hot topic in the literature. For single-mode waveguides with the surface scattering the problem is equivalent to the 1D bulk scattering. For this reason, the methods and results obtained for the latter case can be directly applied for the waveguides [7–9]. Specifically it was shown, both analytically and by direct numerical simulations, that single-mode waveguides with a desired selective transport can be fabricated by a proper construction of long-range-correlated random surfaces.

The key ingredient of the theory of correlated disorder is the binary correlator of a random potential. As was shown, for a weak potential this correlator fully determines the transmission/reflection of waves. The algorithm proposed in Refs. [1,2,4] generates a statistical ensemble of random potentials having the same binary correlator. Note that in this method the values of the potentials are determined by a continuous distribution function. On the other hand, in some applications it is more appropriate to generate a correlated potential from a finite number of elements. An important example of such potentials is a sequence of nucleotides in a DNA molecule, where only four elements determine a potential. In this paper we address the mathematical problem of the construction of a dichotomous sequence with a prescribed type of correlations. A priori, it is not clear whether the random sequence of only two elements may have a given binary correlator. To shed light on this problem we concentrate our attention on statistical properties of a binary Markov chain and their relation to the binary correlator. Our main interest in this study is the so-called memory function that was recently analyzed in other applications (see Refs. [10,11] and references therein). We hope that our results may help to extend the theory of correlated disorder to binary potentials.

The structure of this paper is as follows. In next Section 2, we introduce an additive binary Markov chain via the memory function. We show that the memory function can be related to the pair (binary) correlator of a chain. In Section 3, the integral equation for the binary correlator is developed and discussed in detail. Complimentary, in Section 4 the equation for the memory function is derived in its general form. Next Section 5 is devoted to few other forms of the integral equations for the memory function, that can be more suitable in specific cases. As the simplest case, in Section 6 we consider the Markov chain with a white-noise. Then, in Section 7 a bit more complicated situation is discussed, namely, the binary chain with short-range correlations of a general type. Another interesting case of an exponentially decreasing binary correlator is the subject of Section 8. Unexpectedly, in this case the memory function has a very simple, however, a quite specific form. Another modification of an exponential pair correlator, namely, modulated by the cosine-function, is analyzed in Section 9. In next Section 10, we discuss a general case for which the Fourier transform of the pair correlator has single pole in the complex plane of its argument. Finally, in Section 11 we concentrate our attention on a very important and quite complicated case of the step-wise power (SWP) spectrum. This case is of great importance in view of possible applications of our theory to the transport in physical or DNA systems described by the binary potentials.

## 2. Additive binary Markov chain

In what follows, we consider a *homogeneous random binary sequence* of symbols,

$$\varepsilon(n) = \{\varepsilon_0, \varepsilon_1\}, \quad n \in \mathbf{Z} = \dots, -2, -1, 0, 1, 2, \dots \quad (1)$$

To specify an  $N$ -step *Markov chain* we introduce the *conditional probability* function

$$P(\varepsilon(n) = \varepsilon_{0,1} | \varepsilon(n-1), \varepsilon(n-2), \dots, \varepsilon(n-N+1), \varepsilon(n-N)). \quad (2)$$

It is a probability of appearance of one of the two symbols,  $\varepsilon(n) = \varepsilon_0$  or  $\varepsilon_1$ , after a given sequence  $T_{N,n}$  (“word”) of length  $N$ ,

$$T_{N,n} = \varepsilon(n - 1), \varepsilon(n - 2), \dots, \varepsilon(n - N + 1), \varepsilon(n - N). \tag{3}$$

The *additive* Markov chain is defined by the conditional probability function of the form

$$P(\varepsilon(n) = \varepsilon_{0,1} | T_{N,n}) = p_{0,1} + \sum_{r=1}^N F(r) \frac{\varepsilon(n-r) - \overline{\varepsilon(n)}}{\varepsilon_{0,1} - \varepsilon_{1,0}}. \tag{4}$$

Here  $p_0$  and  $p_1$  are the parameters that imply, respectively, the probability of occurring symbol  $\varepsilon_0$  and  $\varepsilon_1$  in the whole sequence. Evidently,  $p_0 + p_1 = 1$ . For a stationary random sequence these values  $p_0$  and  $p_1$  are, in fact, the only values of the probability density of symbols  $\varepsilon(n)$ . Note that the density does not depend on  $n$ , due to statistical homogeneity. Here and below the bar stands for the average along a sequence. In particular,

$$\overline{\varepsilon(n)} = \lim_{M \rightarrow \infty} \frac{1}{2M + 1} \sum_{n=-M}^M \varepsilon(n) = \varepsilon_0 p_0 + \varepsilon_1 p_1. \tag{5}$$

One should stress that in our case this average is equivalent to the ensemble average.

The function  $F(r)$  describes the effect of correlations between the  $n$ th symbol  $\varepsilon(n)$  and  $N$  previous symbols  $\varepsilon(n-r)$  with  $r = 1, \dots, N$ . In the following we refer to  $F(r)$  as to the *memory function*, since it may be associated with a memory about statistical properties of a sequence, necessary to generate random sequences with given characteristics. Due to the evident condition  $0 \leq P(\cdot) \leq 1$ , this function obeys the restriction

$$\sum_{r=1}^N |F(r)| \leq \min(p_0, p_1) / \max(p_0, p_1). \tag{6}$$

Our main interest is in the relevance of the memory function  $F(r)$  to such a statistical characteristic of random sequences as the *binary correlation function*

$$C(r) = \overline{\varepsilon(n)\varepsilon(n+r)} - \overline{\varepsilon(n)}^2. \tag{7}$$

The statistical homogeneity of the random sequence  $\varepsilon(n)$  provides the correlator  $C(r)$  be dependent only on the distance  $r$  between two points  $n$  and  $n+r$ . Moreover, by definition (7),  $C(r)$  is an even function of this distance,  $C(-r) = C(r)$ . From Eqs. (7) and (5), it follows that the variance  $C(0)$  reads

$$C(0) = \overline{\varepsilon^2(n)} - \overline{\varepsilon(n)}^2 = (\varepsilon_1 - \varepsilon_0)^2 p_0 p_1 \quad \text{with } p_0 + p_1 = 1. \tag{8}$$

As was shown in Refs. [10,11], for additive Markov chains the relation between the memory function  $F(r)$  and correlation function  $C(r)$  has the form

$$C(r) = \sum_{r'=1}^N F(r') C(r-r'), \quad r \geq 1. \tag{9}$$

This equation provides a possibility to obtain the memory function  $F(r)$  from the prescribed correlator  $C(r)$  and to construct effectively the Markov chain with the conditional probability  $P(\cdot)$  according to Eq. (4).

For simplicity, below we use the binary correlator  $K(r)$  normalized to the unity at  $r = 0$ ,

$$K(r) = C(r)/C(0). \tag{10}$$

Also, we assume, without the loss of generality, that the length of considered sequences is infinite. As a result, the starting Eq. (9) gets the form,

$$K(r) = \sum_{r'=1}^{\infty} F(r') K(r-r'), \quad r \geq 1, \tag{11}$$

$$K(-r) = K(r), \quad K(0) = 1. \tag{12}$$

Note that here the memory function  $F(r)$  is defined for  $r \geq 1$ , while the correlator  $K(r)$  being an even function, is determined for all values of  $r$ . To avoid mathematical problems related to this fact, we assume that the memory function  $F(r)$  can be analytically continued to negative values of  $r$ , namely,  $F(-r) = F(r)$ . As one can see, we have also to specify the value of  $F(r)$  for  $r = 0$ . In what follows, for simplicity we assume that  $F(0) = 0$ . As a result, we arrive at main equations that shall be analyzed below:

$$K(r) = \sum_{r'=-\infty}^{\infty} K(r-r')F(r') - \sum_{r'=1}^{\infty} K(r+r')F(r'), \quad r \geq 1, \quad (13)$$

$$K(-r) = K(r), \quad K(0) = 1, \quad (14)$$

$$F(-r) = F(r), \quad F(0) = 0. \quad (15)$$

The above equations can be considered as the starting point for two problems. The first one is to obtain the memory function  $F(r)$  by making use of Eq. (13) in which the correlator  $K(r)$  is known. The second problem is the complimentary one, namely, to find the correlator  $K(r)$  for a given memory function  $F(r)$ .

### 3. Equation for binary correlator

It is convenient to write the Fourier representation of the main equation (13). For this, we introduce the Fourier transform  $\mathcal{K}(k)$  of the binary correlator  $K(r)$ , known as the *randomness power spectrum*,

$$K(r) = \frac{1}{\pi} \int_0^{\pi} dk \mathcal{K}(k) \cos(kr), \quad (16)$$

$$\mathcal{K}(k) = 1 + 2 \sum_{r=1}^{\infty} K(r) \cos(kr), \quad (17)$$

$$\mathcal{K}(-k) = \mathcal{K}(k), \quad \mathcal{K}(k + 2\pi) = \mathcal{K}(k). \quad (18)$$

Since the correlator  $K(r)$  is a real and even function of the coordinate  $r$ , its Fourier transform (17) is an even, real and non-negative function of the wave number  $k$ . The condition  $K(r = 0) = 1$  results in the following normalization relation for the power spectrum  $\mathcal{K}(k)$ :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \mathcal{K}(k) = \frac{1}{\pi} \int_0^{\pi} dk \mathcal{K}(k) = 1. \quad (19)$$

Analogously, the expressions for the Fourier transforms of the memory function  $F(r)$  have the form

$$F(r) = \frac{1}{\pi} \int_0^{\pi} dk \mathcal{F}(k) \cos(kr), \quad (20)$$

$$\mathcal{F}(k) = 2 \sum_{r=1}^{\infty} F(r) \cos(kr), \quad (21)$$

$$\mathcal{F}(-k) = \mathcal{F}(k), \quad \mathcal{F}(k + 2\pi) = \mathcal{F}(k). \quad (22)$$

Similarly, the Fourier transform  $\mathcal{F}(k)$  of the memory function  $F(r)$  is also even and real function of the wave number  $k$ . Due to the imposed condition  $F(r = 0) = 0$ , the normalization condition for  $\mathcal{F}(k)$  reads as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \mathcal{F}(k) = \frac{1}{\pi} \int_0^{\pi} dk \mathcal{F}(k) = 0. \quad (23)$$

Now, we shall derive the Fourier transform of relation (13). First, we represent this equation in the following form:

$$\mathcal{H}(k) = 1 + \sum_{r,r'=-\infty}^{\infty} \exp(ikr)K(r-r')F(r') - \sum_{r=-\infty}^{\infty} K(r)F(r) - 2 \sum_{r,r'=1}^{\infty} \cos(kr)K(r+r')F(r'). \quad (24)$$

Now we substitute the memory function  $F(r')$  due to Eq. (20), together with the use of Eq. (17). In this way we arrive at the relation

$$\mathcal{H}(k)\mathcal{F}(k) - 2 \sum_{r,r'=1}^{\infty} \cos(kr)K(r+r')F(r') = \mathcal{H}(k) + A - 1, \quad (25)$$

where the constant  $A$  is defined by

$$A = \sum_{r=-\infty}^{\infty} K(r)F(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \mathcal{H}(k) \mathcal{F}(k). \quad (26)$$

Into the second term of the left-hand side of Eq. (25) we substitute the correlator  $K(r+r')$  in the form of the Fourier integral (16). As a result, we get the integral equation for the power spectrum  $\mathcal{H}(k)$  of the binary correlator

$$[\mathcal{F}(k) - 1]\mathcal{H}(k) - \frac{1}{2\pi} \int_{-\pi}^{\pi} dk' \Upsilon(k, k') \mathcal{H}(k') = A - 1. \quad (27)$$

Here the integral kernel  $\Upsilon(k, k')$  is described as follows:

$$\begin{aligned} \Upsilon(k, k') &= 2 \sum_{r=1}^{\infty} \cos(kr) \exp(-ik'r) \times \sum_{r'=1}^{\infty} F(r') \exp(-ik'r') \\ &= \frac{1}{2} Q(k, k') \frac{1}{\pi} \int_0^{\pi} dk'' \mathcal{F}(k'') Q(k'', k'), \quad k' = \lim_{\varepsilon \rightarrow +0} (k' - i\varepsilon), \end{aligned} \quad (28)$$

and the  $Q$ -functions are defined below by Eq. (34).

Due to a complex nature of  $Q$ -functions, the kernel  $\Upsilon(k, k')$  is also a complex function. Then, in accordance with definition (28), we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \Upsilon(k, k') = \frac{1}{\pi} \int_0^{\pi} dk \Upsilon(k, k') = 0, \quad (29)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk' \Upsilon(k, k') = \frac{1}{\pi} \int_0^{\pi} dk' \Upsilon(k, k') = 0. \quad (30)$$

When integrating Eq. (27) over the wave number  $k$  within the interval  $(-\pi, \pi)$  or  $(0, \pi)$ , its integral term vanishes due to Eq. (29), while the equation itself turns into identity  $A - 1 \equiv A - 1$ . This fact implies that relation (26) automatically satisfies the solution of Eq. (27), independently on the value of  $A$ . Therefore, the constant  $A$  is determined solely by the normalization condition  $K(0) = 1$  or, the same, by Eq. (19) for the power spectrum  $\mathcal{H}(k)$ . Another form of Eq. (27) will be given below by Eq. (53).

#### 4. Equation for memory function

Let us now substitute the memory function (20) into the left-hand side of Eq. (25). In this way we obtain the integral equation for the Fourier transform  $\mathcal{F}(k)$  of the memory function

$$\mathcal{H}(k)\mathcal{F}(k) - \frac{1}{\pi} \int_0^{\pi} dk' \Xi(k, k') \mathcal{F}(k') = \mathcal{H}(k) + A - 1. \quad (31)$$

In this equation the kernel  $\Xi(k, k')$  of the integral operator is described by the expressions

$$\Xi(k, k') = 2 \sum_{r, r'=1}^{\infty} \cos(kr)K(r+r') \cos(k'r') \quad (32)$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} dk'' \mathcal{H}(k'') Q(k, k'') Q(k', k''), \quad (33)$$

where the  $Q$ -functions are defined as

$$Q(k, k'') = -2 \sum_{r=1}^{\infty} \cos(kr) \exp(-ik''r) \quad (34)$$

$$= 1 + i \frac{\sin k''}{\cos k - \cos k''}, \quad k'' = \lim_{\varepsilon \rightarrow +0} (k'' - i\varepsilon). \quad (35)$$

In spite of a complex nature of the  $Q$ -functions, the kernel  $\Xi(k, k')$  is real, even and symmetrical function of both arguments  $k$  and  $k'$

$$\Xi(-k, k') = \Xi(k, -k') = \Xi(k, k') = \Xi(k', k). \quad (36)$$

One can show that the mean value of the function  $Q(k, k'')$  vanishes,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk Q(k, k'') = \frac{1}{\pi} \int_0^{\pi} dk Q(k, k'') = 0. \quad (37)$$

Indeed, integrating Eq. (35) over  $k$ , one can obtain

$$\begin{aligned} \frac{1}{2k_c} \int_{-k_c}^{k_c} \frac{\sin k'' dk}{\cos k - \cos k''} &= \frac{i\pi}{k_c} \Theta(k_c - |k''|) + 2 \sum_{r=1}^{\infty} \frac{\sin(k_c r)}{k_c r} \sin(k'' r) \\ &= \frac{i\pi}{k_c} \Theta(k_c - |k''|) + \frac{1}{k_c} \ln \left| \frac{\sin[(k_c + k'')/2]}{\sin[(k_c - k'')/2]} \right|, \quad 0 < k_c \leq \pi, \quad |k''| \leq \pi. \end{aligned} \quad (38)$$

The first expression for the integral is obtained by a direct integration of the sum in Eq. (34). In the derivation of the second expression we have used the fact that the Heaviside unit-step function  $\Theta(x)$ , for which  $\Theta(x < 0) = 0$  and  $\Theta(x > 0) = 1$ , arises due to the contribution of simple poles  $k = \pm(k'' - i\varepsilon)$ ,  $\varepsilon \rightarrow +0$ . Clearly, this contribution emerges only when the poles are within the integration interval, i.e., if  $|k''| < k_c$ . The principal value (*P. V.*) of the integral in Eq. (38) has the following form:

$$\begin{aligned} P. V. \int_0^{k_c} \frac{\sin a dx}{\cos x - \cos a} &= \frac{1}{2} P. V. \int_0^{k_c} dx \left[ \cot\left(\frac{x+a}{2}\right) - \cot\left(\frac{x-a}{2}\right) \right] \\ &= \ln \left| \frac{\sin[(k_c + a)/2]}{\sin[(k_c - a)/2]} \right|. \end{aligned} \quad (39)$$

One can see that integral (39) vanishes when  $k_c = \pi$ , and the  $Q$ -function property (37) is evident.

Note that Eq. (37) gives rise to the normalization relations for the kernel  $\Xi(k, k')$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \Xi(k, k') = \frac{1}{\pi} \int_0^{\pi} dk \Xi(k, k') = 0, \quad (40)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk' \Xi(k, k') = \frac{1}{\pi} \int_0^{\pi} dk' \Xi(k, k') = 0. \quad (41)$$

As a result, the integration of Eq. (31) over the wave number  $k$  within the interval  $(-\pi, \pi)$  or  $(0, \pi)$  leads to the identity  $A \equiv A$ . This fact implies that relation (26) automatically satisfies the solution of Eq. (31). Therefore,

the constant  $A$  is determined by the initial condition for the memory function  $F(r)$ ,

$$F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \mathcal{F}(k) = \frac{1}{\pi} \int_0^{\pi} dk \mathcal{F}(k) = 0. \quad (42)$$

## 5. Other forms of equation for memory function

In some cases, instead of Eq. (31) it is more convenient to use other forms of the integral equation for the Fourier transform  $\mathcal{F}(k)$  of the memory function. Specifically, expression (32) for the integral kernel  $\Xi(k, k')$  turns out to be ineffective for some kinds of the correlator  $K(r)$ . In this case one should use Eq. (33) and, therefore, deal with complex  $Q$ -functions (34), taking explicitly into account their pole contributions. In what follows, we assume that  $\mathcal{H}(k)$  and  $\mathcal{F}(k)$  are *even, periodic* and *sectionally continuous* functions on real axis  $k$ , see Eqs. (18) and (22).

In accordance with Eq. (35) the product of two  $Q$ -functions can be written as

$$Q(k, k'')Q(k', k'') = 1 + i \frac{\sin k''}{\cos k - \cos k''} + i \frac{\sin k''}{\cos k' - \cos k''} - \frac{\sin^2 k''}{(\cos k - \cos k'')(\cos k' - \cos k'')}. \quad (43)$$

After the substitution of Eq. (43) into definition (33) for the kernel  $\Xi(k, k')$ , one can obtain

$$2\Xi(k, k') = 1 - \mathcal{H}(k) - \mathcal{H}(k') - 2\Xi_0(k, k'). \quad (44)$$

Here the first term arises due to the unit in Eq. (43) and normalization condition (19). The second and third terms are direct consequences of the following relation:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} = i\mathcal{H}(k), \quad |k| \leq \pi. \quad (45)$$

This relation can be obtained from a more general one

$$\frac{1}{2k_c} \int_{-k_c}^{k_c} dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} = \frac{i\pi}{k_c} \Theta(k_c - |k|) \mathcal{H}(k), \quad 0 < k_c \leq \pi, \quad |k| \leq \pi. \quad (46)$$

The integral in Eq. (46) is determined by simple poles  $k'' - i\varepsilon = \pm|k|$ ,  $\varepsilon \rightarrow +0$ , if the poles are within the integration interval, i.e., when  $|k| < k_c$ . The corresponding principal value of the integral vanishes because the integrand is an odd function. Thus, when  $k_c = \pi$ , Eq. (46) reduces to Eq. (45).

One can see that in Eq. (44) a new integral kernel

$$\Xi_0(k, k') = \frac{1}{4\pi} \int_{-\pi}^{\pi} dk'' \mathcal{H}(k'') \frac{\sin^2 k''}{(\cos k - \cos k'')(\cos k' - \cos k'')} \quad (47)$$

is introduced. Similar to the function  $\Xi(k, k')$ , the new kernel  $\Xi_0(k, k')$  is a real, even and symmetrical function with respect to both arguments  $k$  and  $k'$ ,

$$\Xi_0(-k, k') = \Xi_0(k, -k') = \Xi_0(k, k') = \Xi_0(k', k). \quad (48)$$

In line with Eq. (38) taken at  $k_c = \pi$  and Eq. (45), this new kernel  $\Xi_0(k, k')$  satisfies to the following integral properties:

$$\frac{1}{\pi} \int_0^{\pi} dk \Xi_0(k, k') = -\frac{1}{2} \mathcal{H}(k'), \quad (49)$$

$$\frac{1}{\pi} \int_0^{\pi} dk' \Xi_0(k, k') = -\frac{1}{2} \mathcal{H}(k). \quad (50)$$

These relations, together with Eq. (44) and normalization condition (19), provide the properties (40) and (41) of the old kernel  $\Xi(k, k')$ .

Finally, with the use of Eq. (44), the normalization condition (42) for the memory function, and definition (26) for the constant  $A$ , we arrive at the new equivalent integral equation for the memory function in

$k$ -representation

$$\mathcal{H}(k)\mathcal{F}(k) + \frac{1}{\pi} \int_0^\pi dk' \Xi_0(k, k') \mathcal{F}(k') = \mathcal{H}(k) + \frac{1}{2}A - 1. \quad (51)$$

As one should expect, when integrating Eq. (51) over the wave number  $k$  within the interval  $(-\pi, \pi)$  or  $(0, \pi)$ , its integral term gives the value  $-A/2$  due to the equality (49), thus, reducing the equation into the identity,  $A/2 \equiv A/2$ .

Now let us derive one more form of the integral equation. To this end, one should substitute expression (47) into the integral term of Eq. (51), with a further change of the order of integration over  $k''$  and  $k'$ ,

$$\begin{aligned} & \frac{1}{\pi} \int_0^\pi dk' \Xi_0(k, k') \mathcal{F}(k') \\ &= \frac{1}{4\pi} \int_{-\pi}^\pi dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} \times \frac{1}{\pi} \int_0^\pi dk' \mathcal{F}(k') \frac{\sin k''}{\cos k' - \cos k''} \\ &= \frac{1}{4\pi} \int_{-\pi}^\pi dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} \times \left[ i\mathcal{F}(k'') + \frac{1}{\pi} P.V. \int_0^\pi dk' \mathcal{F}(k') \frac{\sin k''}{\cos k' - \cos k''} \right] \\ &= -\frac{1}{2} \mathcal{H}(k)\mathcal{F}(k) + \frac{1}{4\pi} \int_{-\pi}^\pi dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} \\ & \quad \times \frac{1}{\pi} P.V. \int_0^\pi dk' \mathcal{F}(k') \frac{\sin k''}{\cos k' - \cos k''}. \end{aligned} \quad (52)$$

Here we have employed the ideas used to evaluate integral (38), and formula (45). After we substitute result (52) into Eq. (51), the integral equation reads

$$\begin{aligned} & \mathcal{H}(k)\mathcal{F}(k) + \frac{1}{2\pi^2} \int_{-\pi}^\pi dk'' \mathcal{H}(k'') \frac{\sin k''}{\cos k - \cos k''} \times P.V. \int_0^\pi dk' \mathcal{F}(k') \frac{\sin k''}{\cos k' - \cos k''} \\ &= 2\mathcal{H}(k) + A - 2. \end{aligned} \quad (53)$$

As above, the symbol “ $P.V.$ ” denotes a principal value of the corresponding integral. Since we cannot change the integration order in Eq. (53), its application is suitable if the power spectrum  $\mathcal{H}(k)$  is considered as unknown while the memory function  $\mathcal{F}(k)$  is predefined. When integrating Eq. (53) over  $k$  within the interval  $(-\pi, \pi)$  or  $(0, \pi)$ , its integral term vanishes due to Eq. (38), while the equation itself turns into the identity  $A \equiv A$ .

One should, however, emphasize that all the integral equations (31), (51), and (53) cannot be solved in general case of any form of the additive-Markov-chain power spectrum  $\mathcal{H}(k)$ . This is because of a quite complicated structure of the integral kernels. For this reason, below we shall analyze some particular cases when the solution can be obtained in a relatively simple way.

## 6. White-noise disorder

The white-noise Markov chain is specified by the binary correlator and corresponding power spectrum as follows:

$$K_{wn}(r) = \delta_{r,0}, \quad \mathcal{H}_{wn}(k) = 1. \quad (54)$$

In order to obtain the memory function  $F_{wn}(r)$  of the additive chain, one can use Eqs. (11) or (13). Due to the restriction  $r \geq 1$ , one should substitute zero in their left-hand sides, that gives rise to the expected result

$$F_{wn}(r) = 0. \quad (55)$$

Now, let us see how the same result follows from the integral equations (31), (51) and (53). One should take into account that for a white noise with  $\mathcal{H}_{wn}(k) = 1$  definition (26) for  $A$  and the normalization condition (42) yield  $A = F(0) = 0$ . Therefore, the right-hand side of the equations vanishes and they become homogeneous,

with a trivial solution

$$\mathcal{F}_{wn}(k) = 0. \tag{56}$$

In such a way, we again come to result (55).

Let us write down the integral kernels of the equations with the white-noise power spectrum  $\mathcal{K}_{wn}(k) = 1$ . First, for Eq. (31) we have

$$\begin{aligned} \Xi_{wn}(k, k') &= 2 \sum_{r,r'=1}^{\infty} \cos(kr)\delta_{r+r',0} \cos(k'r') \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} dk'' Q(k, k'')Q(k', k'') = 0. \end{aligned} \tag{57}$$

In this expression the sum is evidently equal to zero. To make sure explicitly that the integral also vanishes, one should employ Eqs. (43), (45) and the relation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin^2 k'' dk''}{(\cos k - \cos k'')(\cos k' - \cos k'')} = -1. \tag{58}$$

As for Eq. (51), taking into account Eq. (58), we get

$$\Xi_0^{wn}(k, k') = -\frac{1}{2}. \tag{59}$$

### 7. Short-range correlations

The simplest non-trivial case for which one can obtain an asymptotically exact solution of Eq. (31), is the additive Markov chain with *short-range correlations*. In this case the correlator  $K(r)$  is assumed to be a rapidly decreasing function of the index  $r$ , with a “very short” scale  $R_c > 0$  of its decrease. It is evident that for short-range correlations the power spectrum  $\mathcal{K}(k)$  is a very smooth function of the wave number  $k$  with a “long” scale  $k_c = R_c^{-1} > 0$  of decrease. One should take into account a discrete nature of the argument  $r$  of the correlator  $K(r)$ , or, the same, the periodicity (18) of its power spectrum  $\mathcal{K}(k)$ . Then, it becomes clear that depending on a specific form of  $K(r)$  (or  $\mathcal{K}(k)$ ), the definition of short-range correlations assumes one of two conditions

$$R_c = k_c^{-1} \ll 1 \quad \text{or} \quad \pi - k_c \ll \pi. \tag{60}$$

In this case the kernel  $\Xi(k, k')$  of the integral operator, see Eq. (32), is mainly contributed by values  $r = r' = 1$ . Consequently, one can get the following estimate for  $\Xi(k, k')$ :

$$\Xi(k, k') \approx 2K(2) \cos k \cos k' \ll K(0) = 1. \tag{61}$$

Therefore, the integral term in Eq. (31) can be neglected. As a result, the integral equation transforms into an algebraic one with the solution

$$\mathcal{F}(k) = 1 - (1 - A)\mathcal{K}^{-1}(k). \tag{62}$$

By making use of Eqs. (20) and (62), the memory function in the coordinate representation reads

$$F(r) = \delta_{r,0} - (1 - A)I(r) \tag{63}$$

with the integral  $I(r)$  defined by

$$I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{\exp(-ikr)}{\mathcal{K}(k)} = \frac{1}{\pi} \int_0^{\pi} dk \frac{\cos(kr)}{\mathcal{K}(k)}. \tag{64}$$

Applying initial condition (42), one can easily get

$$1 - A = I^{-1}(0). \tag{65}$$

As a result, in the case of short-range correlations we arrive at a very simple expression for the memory function  $F(r)$ ,

$$F(r) = \delta_{r,0} - I(r)/I(0). \quad (66)$$

Evidently, this expression should have a crossover to Eq. (55) for a white-noise disorder. Indeed, if  $\mathcal{K}(k) = 1$ , one easily gets  $I(r) = \delta_{r,0}$  and  $F(r) = 0$ .

## 8. Exponential correlations

Clearly, Eq. (31) can be easily solved if the kernel  $\Xi(k, k')$ , see Eq. (32), is factorized by a product of two terms that separately depend on  $k$  and  $k'$  only.

A particular important example, giving rise to such a form of the kernel  $\Xi(k, k')$ , is the Markov chain with an *exponential correlator*

$$K_e(r) = \exp(-k_c|r|), \quad \mathcal{K}_e(k) = \frac{\sinh k_c}{\cosh k_c - \cos k}. \quad (67)$$

Here  $k_c = R_c^{-1} > 0$  determines the inverse correlation length. The explicit form (67) for the randomness power spectrum  $\mathcal{K}(k)$  can be found with the use of the relation,

$$2 \sum_{r=1}^{\infty} \cos(kr) \exp(-k_c r) = \frac{\sinh k_c}{\cosh k_c - \cos k} - 1 \quad (68)$$

that can be obtained by a direct summation of the geometric progression, after rewriting  $\cos(kr)$  in the Euler form.

In accordance with Eqs. (32), (67), and (68), the kernel of the integral operator  $\Xi(k, k')$  can be written in the factorized form

$$\Xi_e(k, k') = \frac{1}{2}[\mathcal{K}_e(k) - 1][\mathcal{K}_e(k') - 1]. \quad (69)$$

This form provides a possibility to find easily the solution of Eq. (31),

$$\mathcal{F}(k) = \frac{2I(0)}{1 + I(0)} [1 - 1/I(0)\mathcal{K}_e(k)], \quad (70)$$

$$F(r) = \frac{2I(0)}{1 + I(0)} [\delta_{r,0} - I(r)/I(0)], \quad (71)$$

with  $I(r)$  given by definition (64).

Taking into account expression (67) for  $\mathcal{K}(k)$  when evaluating the integral in Eq. (64), one can get

$$I(r) = \coth k_c \delta_{r,0} - (2 \sinh k_c)^{-1} \delta_{|r|,1}. \quad (72)$$

Finally, we come to the expression for the memory function

$$F(r) = \exp(-k_c) \delta_{|r|,1}. \quad (73)$$

It is important that the additive binary Markov chain with an exponential correlator is nothing but a sequence with the one-step memory function. As one can see, the exponential nature of the correlator  $K(r)$  results in a one-step-form of the memory function  $F(r)$ , while its variation scale determines the amplitude only.

For the Markov chain of symbols  $\{\varepsilon_0 = 0, \varepsilon_1 = 1\}$  with the relative number  $p_1$  of “1” in the chain, the conditional probability of the symbol “1” occurring after the symbol “1” is equal to

$$P(\varepsilon(i) = 1 | 1) = p_1 + \exp(-k_c)(1 - p_1). \quad (74)$$

Here we took into account that  $\langle \varepsilon(i) \rangle = p_0 \times 0 + p_1 \times 1 = p_1$ . In the limit of an anomalously small correlation length  $k_c^{-1} \rightarrow 0$ , the conditional probability function tends to constant value,  $P(\varepsilon(n) = 1 | 1) \rightarrow p_1$  that is independent of previous symbols. This limit evidently corresponds to the uncorrelated sequence of the type (54). In the opposite case of long-range correlations,  $k_c^{-1} \rightarrow \infty$ , the conditional probability function is almost equal to unity,  $P(\varepsilon(n) = 1 | 1) \rightarrow 1$ . In this case we have a sequence with a maximal persistent diffusion.

There is another important example for which the kernel of the operator  $\Xi(k, k')$  has a separable form. Namely, this is the Markov chain with an exponential *alternating-sign correlator*

$$K(r) = (-1)^r \exp(-k_c|r|), \quad \mathcal{K}(k) = \frac{\sinh k_c}{\cosh k_c + \cos k}. \quad (75)$$

For this case, we obtain the following integral  $I(r)$ :

$$I(r) = \coth k_c \delta_{r,0} + (2 \sinh k_c)^{-1} \delta_{|r|,1}, \quad (76)$$

and, correspondingly, the negative memory function,

$$F(r) = \frac{2I(0)}{1 + I(0)} [\delta_{r,0} - I(r)/I(0)] = -\exp(-k_c) \delta_{|r|,1}. \quad (77)$$

As is known [10], the negative memory function describes the anti-persistent correlated diffusion in the additive binary Markov chain.

To conclude, we would like to note that in the limit of short-range correlations with  $k_c \gg 1$ , one gets  $I(0) \approx 1$ . This leads the exact solutions (73) and (77) that coincide with the asymptotic expression (66).

### 9. Exponential-cosine correlations

In previous Section 8 we have revealed that the exponential binary correlator (67) results in the one-step memory function (73). In order to have a many-steps memory function, we consider the oscillating correlator with an exponential decrease of its amplitude

$$K_{ec}(r) = \exp(-k_0|r|) \cos(\omega r) = \frac{1}{2}[K_e(r) + K_e^*(r)], \quad (78)$$

$$\mathcal{K}_{ec}(k) = \frac{(\cosh k_0 - \cos \omega \cos k) \sinh k_0}{(\cosh k_0 \cos \omega - \cos k)^2 + \sinh^2 k_0 \sin^2 \omega} \quad (79)$$

$$= \frac{1}{2}[\mathcal{K}_e(k) + \mathcal{K}_e^*(k)]. \quad (80)$$

Here the asterisk “\*” stands for the complex conjugation. The last expressions (78) and (80) are given to relate the exp-cosine correlator and its power spectrum to those (67) for purely exponential correlations. Based on this relationship, one should perform the following complex substitution:

$$k_c = k_0 + i\omega. \quad (81)$$

In line with definition (32) and representation (78), the kernel  $\Xi(k, k')$  of the integral operator is described by the sum

$$\Xi_{ec}(k, k') = \frac{1}{2}[\Xi_e(k, k') + \Xi_e^*(k, k')]. \quad (82)$$

Therefore, according to the factorized form (69) of the operator  $\Xi_e(k, k')$ , the integral equation (31) gives rise to the following solution for the memory function:

$$\mathcal{F}_{ec}(k) = \frac{2 + A}{2} - \frac{2 - A}{2} \frac{1}{\mathcal{K}_{ec}(k)} + \frac{B}{2} \frac{\mathcal{K}_e(k) - \mathcal{K}_e^*(k)}{2\mathcal{K}_{ec}(k)}, \quad (83)$$

$$F_{ec}(r) = \frac{2 + A}{2} \delta_{r,0} - \frac{2 - A}{2} I(r) + \frac{B}{2} \Phi(r). \quad (84)$$

Here the real constant  $A$  is defined by the standard expression (26), while the imaginary constant  $B$  is defined by

$$B = \frac{1}{\pi} \int_0^\pi dk \frac{\mathcal{K}_e(k) - \mathcal{K}_e^*(k)}{2} \mathcal{F}_{ec}(k). \quad (85)$$

For real function  $I(r)$  the standard definition (64) remains valid, and new imaginary function  $\Phi(r)$  has the form

$$\Phi(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{\mathcal{H}_e(k) - \mathcal{H}_e^*(k)}{2\mathcal{H}_{ec}(k)} \exp(-ikr), \quad \Re\Phi(r) = 0,$$

$$\frac{\mathcal{H}_e(k) - \mathcal{H}_e^*(k)}{2} = i \frac{(\cos \omega - \cosh k_0 \cos k) \sin \omega}{(\cosh k_0 \cos \omega - \cos k)^2 + \sinh^2 k_0 \sin^2 \omega}. \quad (86)$$

In order to find the values of constants  $A$  and  $B$ , one should write down two equations. The first one is nothing but the normalization condition (42). The second equation can be obtained by substitution of solution (83) into definition (85). As a result, we come to the system

$$[I(0) + 1]A + \Phi(0)B = 2[I(0) - 1], \quad \Phi(0)A - CB = 2\Phi(0), \quad (87)$$

where the real constant  $C$  is given by the expressions

$$C = 2 - \frac{1}{2\pi} \int_0^{\pi} dk \frac{[\mathcal{H}_e(k) - \mathcal{H}_e^*(k)]^2}{2\mathcal{H}_{ec}(k)} = 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{\mathcal{H}_e(k)\mathcal{H}_e^*(k)}{\mathcal{H}_{ec}(k)}. \quad (88)$$

The solution of system (87) is

$$A = 2 \frac{I(0) - 1 + \Phi^2(0)/C}{I(0) + 1 + \Phi^2(0)/C}, \quad B = - \frac{4\Phi(0)/C}{I(0) + 1 + \Phi^2(0)/C}. \quad (89)$$

Thus, the memory function  $F(r)$  of the additive Markov chain with exp-cosine correlations has the form

$$F_{ec}(r) = \frac{2}{I(0) + 1 + \Phi^2(0)/C} \left\{ \left[ I(0) + \frac{\Phi^2(0)}{C} \right] \delta_{r,0} - I(r) - \frac{\Phi(0)\Phi(r)}{C} \right\}. \quad (90)$$

One can see that for  $\omega = 0$  the power spectrum is  $\mathcal{H}_e(k) = \mathcal{H}_e^*(k) = \mathcal{H}_{ec}(k)$ , so that we have  $\Phi(r) = 0$ , and Eq. (90) coincides with Eq. (71).

## 10. Power spectrum with simple complex poles

It is evident that the Fourier integrals (16) and (20) in the definitions of the binary correlator  $K(r)$  and memory function  $F(r)$  are completely determined by singularities of corresponding Fourier transforms  $\mathcal{H}(k)$  and  $\mathcal{F}(k)$  in the complex plane of  $k$ . Taking into account the periodicity conditions (18) and (22), it is clear that the considered Fourier integrals are determined by the singularities located within the complex strip  $(-\pi, \pi)$ .

Let us consider the integral kernel  $\Xi(k, k')$  represented by Eq. (33). It is clear that  $Q$ -functions in the integrand are analytical functions of the integration variable  $k''$  in the lower half-plane of  $k''$ . Indeed, they have periodically repeated simple poles  $k'' = \pm|k| + i\varepsilon$  and  $k'' = \pm|k'| + i\varepsilon$  ( $\varepsilon \rightarrow +0$ ) solely in the upper half-plane and converge to zero when  $k'' \rightarrow -i\infty$ , due to the expression,

$$Q(k, k'')Q(k', k'') \approx 4 \cos k \cos k' \exp(-2y) \quad \text{for } k'' = -iy, \quad y \rightarrow \infty. \quad (91)$$

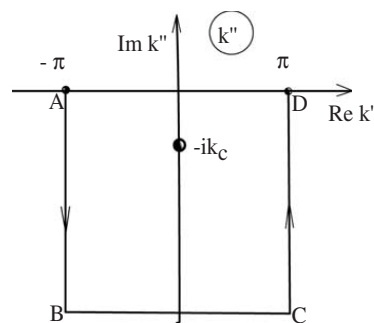


Fig. 1. The contour of integration ABCD in the case of one singularity in the strip.

Therefore, the integral in Eq. (33) is essentially specified by the analytical properties of the power spectrum  $\mathcal{H}(k)$  in the lower half-plane. Let us evaluate the integral along the closed contour  $ABCD$  with  $A = -\pi - i0$ ,  $B = -\pi - i\infty$ ,  $C = \pi - i\infty$  and  $D = \pi - i0$  (see Fig. 1). The sum of integrals along the straight lines  $AB$  and  $CD$  is equal to zero since the integrand is a periodic function with the period  $2\pi$ . Then, the integral along the infinitely far line  $BC$  also vanishes because the product of two last factors of the integrand vanishes exponentially fast at any infinitely far point in the lower half-plane, see Eq. (91). Thus, integral (33) taken with the opposite sign, is completely determined by the singularities of  $\mathcal{H}(k)$  inside the considered contour  $ABCD$ , or the same, within the lower part of the complex strip  $(-\pi, \pi)$ .

Let the power spectrum  $\mathcal{H}(k)$  be analytical within the considered strip. In this case  $\mathcal{H}(k)$  is a constant and due to the normalization condition (19), it has to be one,  $\mathcal{H}(k) = 1$ . From our analysis it follows,  $\Xi(k, k') = 0$ , that is in an accordance with the previously obtained results for a white-noise disorder, see Section 6.

Now we are in a position to analyze the case when there is only one singularity within the strip. Specifically, we consider a simple pole at  $k'' = -ik_c$ , assuming that the function  $\mathcal{H}(k)$  can be represented in the vicinity of this pole by the form

$$\mathcal{H}(k'') \simeq \frac{i}{k'' + ik_c}. \tag{92}$$

Here we take into account that the normalization condition (19) requires the residue be equal to  $i$ . In this case one can get

$$\Xi_e(k, k') = \frac{1}{2}Q(k, -ik_c)Q(k', -ik_c). \tag{93}$$

One can see that the above expression coincides with Eq. (69), thus reducing our problem to that considered in Section 8.

## 11. SWP spectrum

### 11.1. General expressions

As was mentioned in Section 1, in the theory of correlated disorder [1–4], one of the important problems is an additive binary Markov chain with a SWP spectrum. The simplest case is given by the following expressions:

$$K(r) = h\delta_{r,0} + (1-h)\frac{\sin(k_c r)}{k_c r}, \tag{94}$$

$$\mathcal{H}(k) = h + (1-h)\frac{\pi}{k_c}[\Theta(k + k_c) - \Theta(k - k_c)] = h + (1-h)\Delta_{k_c}(k), \tag{95}$$

$$\mathcal{H}^{-1}(k) = \frac{1}{h} \left[ 1 - \frac{k_c(1-h)}{k_c h + \pi(1-h)} \Delta_{k_c}(k) \right], \quad 0 \leq h \leq 1, \quad 0 < k_c \leq \pi, \quad |k| \leq \pi. \tag{96}$$

The restriction on the parameter  $h$  is a consequence of a non-negative nature of the power spectrum  $\mathcal{H}(k)$ . Strictly speaking, the exact requirement is  $0 \leq h \leq \pi/(\pi - k_c)$ . In our analysis we assume  $h$  be smaller than one, in order to allow  $k_c$  be very small.

In Eqs. (95) and (96) the unit-pulse function  $\Delta_{k_c}(k)$  is defined by

$$\Delta_{k_c}(k) = \frac{\pi}{k_c} \Theta(k_c - |k|), \quad 0 < k_c \leq \pi, \quad |k| \leq \pi, \tag{97}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \Delta_{k_c}(k) = \frac{1}{\pi} \int_0^{\pi} dk \Delta_{k_c}(k) = 1, \tag{98}$$

$$\lim_{k_c \rightarrow 0} \Delta_{k_c}(k) = \lim_{k_c \rightarrow 0} \frac{\pi}{k_c} \Theta(k_c - |k|) = 2\pi\delta(k). \tag{99}$$

This definition and hence the power spectrum  $\mathcal{H}(k)$ , together with its inverse quantity  $\mathcal{H}^{-1}(k)$ , contains  $\Theta(x)$  as the Heaviside unit-step function,  $\Theta(x < 0) = 0$  and  $\Theta(x > 0) = 1$ . The factor  $\pi/k_c$  in Eqs. (95) and (97) provides the normalization condition (19). As the majority of unit-pulse functions, our  $\Delta_{k_c}(k)$  is also a prelimit Dirac delta-function. This fact is displayed by Eq. (99). The Fourier transforms for  $\Delta_{k_c}(k)$  are

$$\Delta_{k_c}(k) = \sum_{r=-\infty}^{\infty} \frac{\sin(k_c r)}{k_c r} \exp(ikr) = 1 + 2 \sum_{r=1}^{\infty} \frac{\sin(k_c r)}{k_c r} \cos(kr), \quad (100)$$

$$\frac{\sin(k_c r)}{k_c r} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \Delta_{k_c}(k) \exp(-ikr) = \frac{1}{\pi} \int_0^{\pi} dk \Delta_{k_c}(k) \cos(kr). \quad (101)$$

Note that Eq. (100) for  $\Delta_{k_c}(k)$  coincides in form with the Fourier transforms (16) for  $\mathcal{H}(k)$  and  $K(r)$ . This is not surprising since

$$K(r) = \sin(k_c r)/(k_c r), \quad \mathcal{H}(k) = \Delta_{k_c}(k) \text{ for } h = 0. \quad (102)$$

Therefore, one can expect that all formulas derived in this section have to be reduced to the corresponding general ones in the limit when  $h = 0$ , with the change,  $\Delta_{k_c}(k) \rightarrow \mathcal{H}(k)$ .

One can see that in the case when  $k_c \rightarrow 0$ , we have the sequence with long-range correlations, whereas if  $k_c \rightarrow \pi$  the system is a short-range correlated and at  $k_c = \pi$  it reduces to the white-noise (54). More generally, when  $k_c$  diverges, ( $k_c \rightarrow \infty$ ), the second term in Eq. (94) decreases with oscillations. At the same time,  $K(r)$  gets the white-noise form (54) every time when  $k_c = n\pi$  ( $n = 1, 2, 3, \dots$ ). We emphasize that Eqs. (95)–(96) are applicable only when  $k_c \leq \pi$ . Note that the crossover to the white-noise (54) can be also performed with  $h \rightarrow 1$ .

In order to obtain the memory function  $F(r)$  for the Markov chain with the SWP spectrum (95), one should firstly derive the kernel  $\Xi(k, k')$  of the integral equation (31). For this, we substitute  $K(r + r')$  or  $\mathcal{H}(k')$  in forms (94) or (95) into definitions (32) or (33), respectively. The first term with  $h$  vanishes due to condition (57), therefore, we have

$$\Xi(k, k') = \frac{1-h}{2} 4 \sum_{r, r'=1}^{\infty} \cos(kr) \frac{\sin[k_c(r+r')]}{k_c(r+r')} \cos(k'r') \quad (103)$$

$$= \frac{1-h}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} dk'' \Delta_{k_c}(k'') Q(k, k'') Q(k', k'') \quad (104)$$

$$= \frac{1-h}{2} \frac{1}{2k_c} \int_{-k_c}^{k_c} dk'' Q(k, k'') Q(k', k''). \quad (105)$$

Now we employ Eq. (43) for the product of two  $Q$ -functions to write explicitly some integrals. This yields

$$\Xi(k, k') = \frac{1-h}{2} [1 - \Delta_{k_c}(k) - \Delta_{k_c}(k') + I_{sw}(k, k')]. \quad (106)$$

Here the new integral kernel reads

$$I_{sw}(k, k') = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk'' \Delta_{k_c}(k'') \times \frac{\sin^2 k''}{(\cos k - \cos k'')(\cos k' - \cos k'')} \quad (107)$$

$$= -\frac{1}{2k_c} \int_{-k_c}^{k_c} \frac{\sin^2 k'' dk''}{(\cos k - \cos k'')(\cos k' - \cos k'')} \quad (108)$$

$$= 1 - \frac{1}{k_c} \frac{1}{\cos k - \cos k'} \left[ \sin k \ln \left| \frac{\sin[(k_c + k)/2]}{\sin[(k_c - k)/2]} \right| - \sin k' \ln \left| \frac{\sin[(k_c + k')/2]}{\sin[(k_c - k')/2]} \right| \right]. \quad (109)$$

When  $k_c = \pi$ , we have  $I_{sw}(k, k') = 1$ , and the kernel  $\Xi(k, k')$  vanishes. This fact is in an agreement with condition (57). Actually, the same expression (106) for the kernel  $\Xi(k, k')$  can be achieved from Eq. (44) by taking explicitly into account definition (47) and Eq. (58).

The kernel  $I_{sw}(k, k')$  is a real, even and symmetrical function of both variables  $k$  and  $k'$

$$I_{sw}(-k, k') = I_{sw}(k, -k') = I_{sw}(k, k') = I_{sw}(k', k). \quad (110)$$

In line with Eq. (38) taken at  $k_c = \pi$  and Eq. (46), this new kernel  $I_{sw}(k, k')$  satisfies the following integral properties:

$$\frac{1}{\pi} \int_0^\pi dk I_{sw}(k, k') = \Delta_{k_c}(k'), \tag{111}$$

$$\frac{1}{\pi} \int_0^\pi dk' I_{sw}(k, k') = \Delta_{k_c}(k). \tag{112}$$

These relations together with Eq. (106) and normalization condition (98) provide the integral properties (40) and (41) of the old kernel  $\Xi(k, k')$ .

Now we substitute Eqs. (95) and (106) into Eq. (31). One should take into account that two first summands from  $\Xi(k, k')$  do not contribute into the integral term of Eq. (31), due to the normalization condition (42). As a result, one can get the integral equation for the memory function

$$\frac{h}{1-h} \mathcal{F}(k) + \Delta_{k_c}(k) \mathcal{F}(k) - \frac{1}{2\pi} \int_0^\pi dk' I_{sw}(k, k') \mathcal{F}(k') = \Delta_{k_c}(k) - 1 + \frac{1}{2} B_{sw}. \tag{113}$$

Here, instead of the old constant  $A$ , for the convenience we have introduced the new one,  $B_{sw}$ , which can be associated with  $A$  according to Eq. (26)

$$A = (1-h)B_{sw}, \tag{114}$$

$$B_{sw} = \frac{1}{2\pi} \int_{-\pi}^\pi dk \Delta_{k_c}(k) \mathcal{F}(k) = \frac{1}{2k_c} \int_{-k_c}^{k_c} dk \mathcal{F}(k). \tag{115}$$

Note that Eq. (113) can be derived from Eq. (51).

Finally, let us integrate Eq. (113) over the wave number  $k$  within the interval  $(-\pi, \pi)$  or  $(0, \pi)$ . Then, on its left-hand side the first term vanishes due to Eq. (42), second term turns into  $B_{sw}$  in accordance with its definition (115), while the last integral term gives the value  $B_{sw}/2$  because of the property defined by Eqs. (111) and (115). On the right-hand side, the first term compensates the unity. In such a manner, the equation is reduced to the identity  $B_{sw}/2 \equiv B_{sw}/2$ .

It should be, however, stressed that having Eq. (113) in its general form, it is still not clear how to derive the solution.

### 11.2. SWP spectrum: short-range correlations

A particular case is the SWP spectrum with short-range correlations, which corresponds to the values of  $k_c$  close to  $\pi$ . In this case Eqs. (94) and (95) result in a particular case of Eq. (66) with the integral  $I(r)$  that can be calculated in accordance with its definition (64). Taking into account Eq. (96) one can derive

$$I(r) = \frac{1}{h} \left[ \delta_{r,0} - \frac{k_c(1-h)}{\pi - (\pi - k_c)h} \frac{\sin(k_cr)}{k_cr} \right]. \tag{116}$$

Therefore, the memory function has the form

$$F(r) = \frac{k_c(1-h)}{(\pi - k_c)(1-h) + k_ch} \left[ \frac{\sin(k_cr)}{k_cr} - \delta_{r,0} \right] \quad \text{for } \pi - k_c \ll \pi. \tag{117}$$

As a result, within the first approximation in  $h \rightarrow 0$  we get

$$F(r) = \frac{k_c}{\pi - k_c} \left[ \frac{\sin(k_cr)}{k_cr} - \delta_{r,0} \right], \tag{118}$$

which is valid for  $k_ch \ll \pi - k_c \ll \pi$ .

### 11.3. SWP spectrum: long-range correlations (zero approximation)

Now let us consider the more interesting case of the SWP spectrum (95) with long-range correlations. In this case the additive binary Markov chain has small wave number  $k_c \ll 1$ . Thus, within zero approximation in  $k_c$  the exact expressions (95) for the binary correlator  $K(r)$  and its power spectrum  $\mathcal{K}(k)$  can be changed by the following asymptotics:

$$K(r) = h \delta_{r,0} + (1 - h), \quad (119)$$

$$\mathcal{K}(k) = h + (1 - h)2\pi\delta(k). \quad (120)$$

with  $|k| \leq \pi$ . Note that these expressions are in an agreement with Eqs. (97)–(99) and (100)–(101).

Now we substitute  $K(r + r')$  or  $\mathcal{K}(k')$  in the form of Eqs. (119) or (120) into the definitions, respectively, (32) or (33) for the kernel  $\Xi(k, k')$ . Taking into account that the first term with  $h$  vanishes due to condition (57), we get the result

$$\Xi(k, k') = \frac{1-h}{2} \left[ 2 \sum_{r=1}^{\infty} \cos(kr) \right] \left[ 2 \sum_{r'=1}^{\infty} \cos(k'r') \right] \quad (121)$$

$$= \frac{1-h}{2} Q(k, 0)Q(k', 0) \quad (122)$$

$$= \frac{1-h}{2} [1 - 2\pi\delta(k)][1 - 2\pi\delta(k')] \quad (123)$$

$$= \frac{1-h}{2} [1 - 2\pi\delta(k) - 2\pi\delta(k') + 4\pi^2\delta(k)\delta(k')], \quad (124)$$

that should be compared with the exact expressions (103)–(105) and (106). In order to pass from Eqs. (121) and (122) to Eq. (123) and then to Eq. (124), one should recognize that from definition (34) for the  $Q$ -function, it follows:

$$Q(k, 0) = -2 \sum_{r=1}^{\infty} \cos(kr) = 1 - 2\pi\delta(k). \quad (125)$$

Note that the expressions (121)–(124) and (125) fulfil the general relations (40)–(41).

The integral equation (31) with Eq. (124) taken as the kernel  $\Xi(k, k')$ , gives rise to the equation

$$\frac{h}{1-h} \mathcal{F}(k) + 2\pi\delta(k)\mathcal{F}(k) - \pi\delta(k)\mathcal{F}(0) = 2\pi\delta(k) - 1 + \frac{1}{2} \mathcal{F}(0). \quad (126)$$

Here we take into account that

$$B_{sw} = \mathcal{F}(0) \quad \text{for } k_c \rightarrow 0. \quad (127)$$

All terms in Eq. (126) correspond to those in the general equation (113). Thus, the solution for the memory function yields

$$\mathcal{F}(k) = \frac{1-h}{h} \left[ 1 - \frac{1}{2} \mathcal{F}(0) \right] [2\pi\delta(k) - 1], \quad (128)$$

$$F(r) = \frac{1-h}{h} \left[ 1 - \frac{1}{2} \mathcal{F}(0) \right] [1 - \delta_{r,0}], \quad k_c \rightarrow 0. \quad (129)$$

In principle, these relations would give a complete solution of the problem. However, a problem remains how to find the constant  $B_{sw} = \mathcal{F}(0)$ . Indeed, the solution (128)–(129) automatically satisfies to the normalization condition (42). On the other hand, relation (128) violates if one takes  $k = 0$ .

11.4. SWP spectrum: long-range correlations (another approach)

Our failure to derive the correct solution in previous subsection is evidently related to the fact that at the beginning we have substituted the rigorous expressions (94)–(95) by Eqs. (119)–(120). Strictly speaking, the latter are always invalid since the smaller  $k_c$  the larger indices  $r \sim k_c^{-1}$  play an important role in Eq. (94) and in the sum of Eq. (103). This results in the value of  $\Delta_{k_c}(k)$ , not in  $2\pi\delta(k)$ , in expression (95) for the power spectrum and (104) for the integral kernel, see also Eqs. (100)–(101).

In order to improve the approach formulated above, let us replace, respectively, the Dirac delta-functions  $2\pi\delta(k)$  and  $2\pi\delta(k')$  with prelimit ones  $\Delta_{k_c}(k)$  and  $\Delta_{k_c}(k')$  in asymptotical Eqs. (123) and (124). With this, the expression for the kernel turns out to have the form

$$\Xi(k, k') = \frac{1-h}{2} 4 \sum_{r, r'=1}^{\infty} \cos(kr) \frac{\sin(kcr)}{kcr} \times \frac{\sin(kc r')}{kcr'} \cos(k' r') \tag{130}$$

$$= \frac{1-h}{2} [1 - \Delta_{k_c}(k)][1 - \Delta_{k_c}(k')]. \tag{131}$$

Let us compare the first expression (130) with *exact* Eq. (103) and zero-asymptotical Eq. (121). In the *exact* expression (106) for the kernel  $\Xi(k, k')$  in the case of long-range correlations,  $k_c \ll 1$ , we admit that the following replacement:

$$I_{sw}(k, k') \rightarrow \Delta_{k_c}(k)\Delta_{k_c}(k') \tag{132}$$

is correct. Note that such a model conserves all general properties of the function  $I_{sw}(k, k')$  described above after Eqs. (107)–(108), and as a consequence, provides all properties of the kernel  $\Xi(k, k')$ , as well as the exact equations (31) and (113).

Assumption (132) allows one straightforwardly to obtain the solution of Eq. (113)

$$\frac{2h}{1-h} \mathcal{F}(k) = (2 + B_{sw}) \left[ 1 + \frac{1-h}{h} \Delta_{k_c}(k) \right]^{-1} \Delta_{k_c}(k) - (2 - B_{sw}) \left[ 1 + \frac{1-h}{h} \Delta_{k_c}(k) \right]^{-1}. \tag{133}$$

Now one should take into account the equalities

$$\left[ 1 + \frac{1-h}{h} \Delta_{k_c}(k) \right]^{-1} = 1 - \frac{k_c(1-h)}{k_ch + \pi(1-h)} \Delta_{k_c}(k), \tag{134}$$

$$\left[ 1 + \frac{1-h}{h} \Delta_{k_c}(k) \right]^{-1} \Delta_{k_c}(k) = \frac{k_ch}{k_ch + \pi(1-h)} \Delta_{k_c}(k), \quad \Delta_{k_c}^2(k) = \frac{\pi}{k_c} \Delta_{k_c}(k). \tag{135}$$

Then the solution gets the form

$$\frac{2h}{1-h} \mathcal{F}(k) = -(2 - B_{sw}) + \frac{4k_ch - (2 - B_{sw})[k_ch - k_c(1-h)]}{k_ch + \pi(1-h)} \Delta_{k_c}(k). \tag{136}$$

From the normalization condition (42) one can obtain

$$2 - B_{sw} = \frac{4k_ch}{(\pi - k_c)(1-h) + 2k_ch}. \tag{137}$$

As a result, in the case of long-range correlations we have

$$\mathcal{F}(k) = \frac{2k_c(1-h)}{(\pi - k_c)(1-h) + 2k_ch} [\Delta_{k_c}(k) - 1], \tag{138}$$

$$F(r) = \frac{2k_c(1-h)}{(\pi - k_c)(1-h) + 2k_ch} \left[ \frac{\sin(kcr)}{kcr} - \delta_{r,0} \right]. \tag{139}$$

Evidently, this expression can be simplified by making use of a smallness of  $k_c$ . Thus, the final correct expression gets the form

$$\mathcal{F}(k) = \frac{2k_c}{\pi} [\Delta_{k_c}(k) - 1], \quad (140)$$

$$F(r) = \frac{2k_c}{\pi} \left[ \frac{\sin(k_c r)}{k_c r} - \delta_{r,0} \right]. \quad (141)$$

Note that the obtained result (140)–(141) is independent of  $h$ , hence it is also valid for  $h = 0$ .

## 12. Concluding remarks

In this paper we suggest a method of construction of a binary Markov chain with a prescribed pair correlator. It is based on the integral relation between the pair correlator and the so-called memory function. The knowledge of the latter allows one to create binary sequences that have given short or long-range pair correlations. Our interest in this method is mainly due to its application to the problem of selective transport in one-dimensional disordered systems. As is known, the theory of quantum transport through finite samples described by a white-noise potential is fully developed. The key ingredient is the well known Anderson localization, that states that all eigenstates in infinite samples are exponentially localized regardless of the strength of disorder. The knowledge of the localization length of these eigenstates allows for a complete statistical description of all transport properties of finite samples, due to the single-parameter scaling conjecture.

In contrast with the white-noise disorder, the role of long-range correlations in random potentials is far from being properly understood. Recent results reveal quite unexpected properties of the transport in systems with such potentials. In particular, in Refs. [1–9] it was shown that specific long-range correlations give rise in a very sharp change of the transmission coefficient, when the energy of an incident wave crosses some value  $E_c$ . This effect is similar to the mobility edge effect in three-dimensional solid state models, according to which on one side of  $E_c$  all eigenstates are localized, in contrast with other side where the eigenstates are extended. It is assumed that analogous effects in one-dimensional geometry may find various applications, such as a construction of materials with a highly selective transport or explanation of a selective conductivity in DNA molecules.

However, as was recently understood, an extension of this theory to binary sequences suffers from unexpected problems. We hope that our study of the integral equation (11) relating the memory function and pair correlator can shed light on the problem of binary correlated disorder. Since there is no general theory for solving integral equations, we have decided to start with particular cases allowing for analytical solutions. In this line, apart from the study of general properties of the main equation (11), we have solved relatively simple, however, physically interesting cases of short-range, exponentially decaying, and exponential-cosine correlations. In all these cases we did not meet serious mathematical problems. On the other hand, a more general problem of the step-function (95) that is of specific interest in view of the mobility edge transition, we were able to solve this problem for the cases when the mobility edge is close to either one or other energy band. We would like to stress that these two cases are also interesting from the view point of different applications, giving close analytical expressions for the pair correlators. We have found, however, that it is not clear how to obtain the general solution of the problem. It is still not clear whether the mathematical difficulties of solving the difference equation (11) are principal, or they can be overcome using some different approaches. Note, that here we discuss the analytical results only, leaving aside a possibility to solve the problem numerically. We plan to do this in our next studies.

To conclude with, we would like to note that the problem of construction of sequences with a power-law decay of the pair correlator, is quite specific. In particular, one should take into account strong restrictions imposed by a rigorous theorem stating that the Fourier transform of any pair correlator is a non-negative function (this corresponds to the fact that the Lyapunov exponent of the considered processes cannot be negative). This results in a quite unexpected conclusions. For example, there are no sequences giving rise to a monotonic power decay for the pair correlator, such as  $K(r) = C/r^\gamma$ , where  $C$  and  $\gamma$  are some constants.

In other words, one should always take into account the oscillations of a pair correlator, such as  $\sin(k_c r)$ . These oscillations play an important role in providing the non-negative values of the Lyapunov exponents (see, for example, Ref. [1]). Therefore, it is not clear whether one can speak about “generic cases” of the power decaying correlations.

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